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On Large Deviations for Sums of Random Vectors in R_k

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The paper deals with limit theorems for probabilities of large deviations for sums of independent identically distributed random vectors. We give more detailed bounds for the remainder in von Bahr's limit theorem. New asymptotic formulas for probabilities of large deviations on the outside of balls are established.

1. INTRODUCTION

Let $\{\xi_n\}$ be a sequence of independent identically distributed random vectors in Euclidean space R_k having zero means and a nonsingular covariance matrix U, $\zeta_n = n^{-1/2} \sum_{i=1}^n \xi_i$. Usually, the problem of large deviations for ζ_n is to investigate the asymptotic behavior of the probability $P\{\zeta_n \in \mathscr{E}_n\}$ for the set $\mathscr{E}_n$ such that this probability goes to 0 as $n \to \infty$. Borovkov and Rogozin [4], von Bahr [2], and Nagaev and Sakojan [5] give the asymptotic formulas for the probability $P\{\zeta_n \in \mathscr{E}_n\}$ for an appropriate class of sets $\mathscr{E}_n$. In particularly, von Bahr treats the class of sets being the difference of two convex Borel sets. In an earlier paper [7] Richter obtains the asymptotics of $P\{\zeta_n \in \mathscr{E}_n\}$ for the special case when $\mathscr{E}_n$ is the outside of the ball centered at zero and the covariance matrix U is unity.

Section 3 of this paper gives a result containing a more detailed bound for the remainder term in von Bahr's theorem [2]. Our bound (Theorem 1) shows the dependence of the remainder upon covariance matrix U and is rough enough; however Theorem 1 allows one to obtain some results on large deviations for the sums ζ_n in the infinite-dimensional Hilbert space [6]. Theorem 1 in Section 4 is used to obtain an approximation to $P\{\zeta_n \in \mathscr{E}_n\}$ for the sequence of the sets $\mathscr{E}_n = \{x \in R_k : |x| > r_n\}$, where $1 < r_n = o(\sqrt{n})$,

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$n \to \infty$. The results of Section 4 extend Richter's theorems [7] to the case when U is not the unity matrix.

For $x, y \in R_k$, (x, y) denotes the usual inner scalar product between x and y, $|x| = \sqrt{(x, x)}$. Throughout the paper, c (or ε) denote positive constants (or small positive constants). Indexation of constants begins a new in each section.

2. Preliminary Lemmas

Let ξ be a random vector in R_k with the distribution F, zero mean and the nonsingular covariance matrix U. Let γ denote the minimal eigenvalue of the covariance matrix U, $\gamma > 0$. We shall suppose that

$$E \exp\{t\,|\xi|\} \leqslant m_0 \qquad \text{for all} \quad t \in [-t_0, t_0], \tag{A}$$

where m_0, t_0 are some positive constants.

Let $S(t)$ $(\overline{S(t)})$ denotes the open (closed) ball in R_k centered at zero, with the radius $t > 0$. Put

$$\varphi(h) = E \exp\{(h, \xi)\}.$$

If condition (A) holds, the functional $\varphi(h)$ is finite for all $h \in \overline{S(t_0)}$. For the distribution $F(\cdot)$ we define the conjugate distribution $F(\cdot, h)$ in R_k by

$$F(dx, h) = \frac{1}{\varphi(h)} \exp\{(h, x)\}\, F(dx), \qquad h \in \overline{S(t_0)}.$$

Let $m(h)$ and $\sigma(h)$ be respectively the mean and the covariance matrix of the conjugate distribution $F(\cdot, h)$.

Consider the equation between the variables $h, v \in R_k$ given by

$$m(h) = v. \tag{2.1}$$

It is known that Eq. (2.1) defines a continuous one-to-one mapping between h and v such that the convex sets map to the same (for details see Borovkov and Rogozin [4]). The solution of Eq. (2.1) will be denoted by $h = h(v)$. We shall need the following elementary lemma on properties of the function $h(v)$.

Lemma 1. *If condition (A) is satisfied, then the solution $h = h(v)$ of Eq.* (2.1) *exist for all $v \in \overline{S(\varepsilon_1 \gamma^2)}$ and*

$$|h(v)| \leqslant 2\gamma^{-1}\,|v|, \qquad v \in \overline{S(\varepsilon_1 \gamma^2)}, \tag{2.2}$$

where the positive constant ε_1 depends only on m_0 and t_0.

Proof. In view of (A) the left-hand side of (2.1) is finite for all $h \in S(t_0)$. In order to prove the first assertion of the lemma we show that the set $m(S(t_0))$ includes the ball of the radius $\varepsilon_1 \gamma^2$. Note that $\gamma < c_1$ (the constant c_1 depends only on m_0 and t_0) and condition (A) is satisfied for all $h \in S(4\varepsilon_1 \gamma)$, where $\varepsilon_1 = t_0(4c_1)^{-1}$. Using the Taylor expansion at the point $h = 0$ and condition (A) we have

$$m(h) = \frac{1}{\varphi(h)} E\xi \exp\{(\xi, h)\} = Uh + \theta_1 h^2, \qquad h \in S(2\varepsilon_1 \gamma), \tag{2.3}$$

where $|\theta_1| < c_2$. Therefore $m(0) = 0$ and

$$m(h) \geqslant |h| \, (\gamma - c_2 |h|) > \tfrac{1}{2}\gamma |h|, \qquad h \in S(2\varepsilon_1 \gamma),$$

if $\varepsilon_1 < (4c_2)^{-1}$. It is easy to show that $m(S(2\varepsilon_1 \gamma)) \supset S(\varepsilon_1 \gamma^2)$ and the second assertion is also true.

Let $\varphi'(h)(x)$ and $\varphi''(h)(x, y)$, $x, y \in R_k$, denote the Frechet derivatives of the order one and two of the functional φ at the point h. Under condition (A) this derivatives exist for all $h \in S(t_0)$. Put

$$\Lambda(h, v) = -(h, v) + \ln \varphi(h), \qquad h \in S(t_0), \quad v \in R_k.$$

LEMMA 2. *If condition* (A) *is satisfied, then for all* $v \in S(\varepsilon_1 \gamma^2)$ *and* $x, y \in R_k$

$$[\Lambda(h(v), v)]'_v (x) = -(h(v), x), \tag{2.4}$$

$$[\Lambda(h(v), v)]''_{vv} (x, y) = -(\sigma^{-1}(h(v))\, x, y), \tag{2.5}$$

$$(\sigma^{-1}(h(v))\, x, x) = (U^{-1}x, x)(1 + \theta_2 \gamma^{-2} |v|), \qquad |\theta_2| < c_3, \tag{2.6}$$

where the constant c_3 *depends only on* m_0 *and* t_0.

Proof. The identities (2.4) and (2.5) can be verified by simple differentiating and using (2.1). Taking into account Taylor's theorem it is easy to show that for all $h \in S(2\gamma\varepsilon_1)$ and $x \in R_k$

$$\sigma(h)\, x = Ux + \theta_3 |h| \, |x|, \qquad |\theta_3| < c_4.$$

Note that the minimal eigenvalue of positive operator $\sigma(h)$ is $\geqslant \gamma(1 - c_4 |h|)$. Moreover we have

$$(\sigma^{-1}(h)\, x, x) = (U[\sigma^{-1}(h)\, x], [U^{-1}x]) = (x, U^{-1}x) + W,$$

where

$$|W| \leqslant c_4 |h| \, |\sigma^{-1}(h) x| \, |U^{-1} x|$$
$$\leqslant c_5 \gamma^{-1} |h| \, [(\sigma^{-1}(h) x, x) + (U^{-1} x, x)].$$

From here and Lemma 1, relation (2.6) follows.

Define the measure in R_k by the following identity:

$$\mu_n(B) = (2\pi)^{-k/2} \det^{-1/2} U \int_B \exp \left\{ n\Lambda \left(h \left(\frac{x}{\sqrt{n}} \right), \frac{x}{\sqrt{n}} \right) \right\} dx,$$
$$B \subset S(\varepsilon_1 \gamma^2 \sqrt{n}),$$

where n is an integer and dx is the element of the volume in R_k. For any Borel set B in R_k and $\varepsilon > 0$ we put $B = \{x: |x - y| < \varepsilon \text{ for some } y \in B\}$.

3. On von Bahr's Theorem

Let ξ, ξ_1, ξ_2,... be a sequence of independent random vectors in R_k with a common distribution F, $E\xi = 0$ and let F_n be the distribution of the normed sum $\zeta_n = n^{-1/2}(\xi_1 + \cdots + \xi_n)$.

THEOREM 1. *Let* $l \in [1, \sqrt{n}]$ and let $B^{(1)}$, $B^{(2)}$ be two convex Borel sets in R_k. If condition (A) holds and $\mathscr{E} = B^{(1)} \backslash B^{(2)} \subset S(\varepsilon_1 \gamma^2 l)$, then

$$F_n(\mathscr{E}) = \mu_n(\mathscr{E}) + \theta_1 \frac{l}{\sqrt{n}} \mu_n(\mathscr{E}_\varepsilon),$$

where

$$|\theta_1| \leqslant c_1 \gamma^{-3/2} \exp\{c_2 \gamma^{-3/2}\}, \qquad \varepsilon = \sqrt{\gamma}\, l^{-1}.$$

The constant ε_1 is the same as in Lemma 1 and the constants c_1, c_2 depend only on k, m_0 and t_0.

The first step in the proof of Theorem 1 is to generalize the bound in the central limit theorem in R_k given by von Bahr [1]. For this aim we use the method of the smoothing distribution presented by Sazonov [8].

For any Borel set B in R_k let $v(B)$ denote the volume of B, ∂B denote the boundary of B and $B_{-\varepsilon} = (B_\varepsilon^c)^c$ (where $B^c = R_k \backslash B$). The symbol $*$ means convolution.

LEMMA 3. *Let W, V be probability distributions in R_k; let ϕ_σ be the Gaussian distribution in R_k with zero mean and a nonsingular covariance matrix σ. Let ε be a positive constant and $a = V(S^c(\varepsilon))$.*

For any Borel set B in R_k we have

$$\begin{aligned}&\sup\{|(W-\phi_\sigma)(B-x)| : x \in R_k\}\\ &\quad\leqslant (1-2\alpha)^{-1} \det{}^{-1/2}\sigma \nu((\partial B)_{2\epsilon})\\ &\qquad + \sup\{|(W-\phi_\sigma) * V(B_t + x)| : t = \pm\varepsilon, x \in R_k\}\end{aligned}$$

whenever $\alpha < \frac{1}{2}$.

Proof. First we state a lower bound for the convolution $\int (W-\phi_\sigma)(B_\epsilon - x)\, V(dx)$. Using the following elementary inequalities for the integrand

$$\begin{aligned}&(W-\phi_\sigma)(B_\epsilon - x)\\ &\quad\geqslant (W-\phi_\sigma)(B) - \phi_\sigma(B_{2\epsilon}\backslash B)\\ &\quad\geqslant (W-\phi_\sigma)(B) - \det{}^{-1/2}\sigma\cdot \nu(B_{2\epsilon}\backslash B), \qquad x \in S(\varepsilon),\end{aligned}$$

and

$$\begin{aligned}&(W-\phi_\sigma)(B_\epsilon - x)\\ &\quad\geqslant (W-\phi_\sigma)(B-x) - \det{}^{-1/2}\sigma\cdot (B_{2\epsilon}\backslash B), \qquad x \notin S(\varepsilon),\end{aligned}$$

we find that

$$\begin{aligned}(W-\phi_\sigma) * V(B_\epsilon) \geqslant (1-\alpha)(W-\phi_\sigma)(B) - \nu(B_{2\epsilon}\backslash B)\det{}^{-1/2}\sigma\\ - \alpha\sup\{|(W-\phi_\sigma)(B+x)| : x \in R_k\}.\end{aligned}$$

LEMMA 4. *Let η, η_1, η_2,... be independent identically distributed random vectors in R_k with zero means, a nonsingular covariance matrix σ and the finite third moment $\rho = E|\eta|^3$. Let W_n be the distribution of the sum $n^{-1/2}(\eta_1 + \cdots + \eta_n)$.*

For any Borel set B in R_k

$$\begin{aligned}&|(W_n - \phi_\sigma)(B)|\\ &\quad\leqslant c_3 \det{}^{-1/2}\sigma[\rho\gamma_\sigma^{-3/2} n^{-1/2}\nu(B_\epsilon) + \nu((\partial B)_{2\epsilon})],\end{aligned}$$

where $\varepsilon = c_4\rho(\gamma_\sigma\sqrt{n})^{-1}$, γ_σ is the minimal eigenvalue of covariance matrix σ and the constants c_3, c_4 depend only on k.

Proof. Take in Lemma 3 $W = W_n$ and the constant ε as above. The smoothing distribution V is defined by $V(dx) = V(T\,dx)$, where $T = (8\rho)^{-1}\gamma_\sigma\sqrt{n}$ and the distribution V has the density

$$q(x) = \alpha_k\left[|x|^{k/2} J_{k/2}\left(\frac{|x|}{6}\right)\right]^6$$

($J_t(x)$ is the Bessel function, α_k is the normed constant). Then the characteristic function $v(t)=\int \exp\{i(t,x)\}\, q(x)\, dx$ of the distribution V is the sixfold convolution of the uniform distribution on the ball $S(\frac{1}{6})$ so that $v(t)=0$ for $|v|>1$. We choose the constant $c_4=c_4(k)$ so large that $\alpha = V(S^c(\frac{1}{8}c_4))<\frac{1}{4}$. By the inversion formula we have

$$(W_n-\phi_\sigma)*V(B_{\pm\epsilon})=(2\pi)^{-k}\int_{R_k}\left\{\int_{B_{\pm\epsilon}}\exp\{-i(t,x)\}\,dx\right\}$$
$$\times\left(f_n(t)-\exp\left(-\frac{1}{2}(\sigma t,t)\right)\right)v\left(\frac{t}{T}\right)dt,$$

where f_n is the characteristic function W_n. It is clear that the inner integral is $\leqslant v(B_\epsilon)$. Now we estimate the external integral by the next lemma.

LEMMA 5 (Bikelis [3]). *If the assumptions of Lemma* 4 *hold, then for all* $t\in S((8\rho)^{-1}\gamma_\sigma\sqrt{n})$

$$|f_n(t)-\exp\{-\tfrac{1}{2}(\sigma t,t)\}|$$
$$\leqslant c_5\rho n^{-1/2}|t|^3\exp\{-\tfrac{1}{4}(\sigma t,t)\},$$

where the constant c_5 *depends only on* k.

Return to the proof of Theorem 1. In the rest of the Section 3 the constants c_s, $s\geqslant 6$, depend only on m_0, t_0 and k. At first we prove that

$$F_n(\mathscr{E})=\mu_n(\mathscr{E})+\theta_2\frac{l}{\sqrt{n}}\mu_n(K),\qquad |\theta_2|<c_6, \tag{3.1}$$

for any set $\mathscr{E}=B^{(1)}\backslash B^{(2)}$ satisfied the conditions of Theorem 1 and the additional condition $\mathscr{E}\subset K$ where K is a k-dimensional cube with the rib of length $f=\sqrt{\gamma}\,(l\sqrt{k})^{-1}$. Let $\bar{a}$ be the centre of the cube K and let $a=\bar{a}n^{-1/2}$. Consider the n-fold normed convolution of the conjugate distribution given by

$$F_n(dx,h)=F^{*n}(\sqrt{n}\,dx+nm(h),h).$$

We have (see [2])

$$F_n(\mathscr{E})=\varphi^n(h)\int_{\mathscr{E}-\bar{a}}\exp\{-\sqrt{n}\,(h,x+\bar{a})\}\,F_n(dx,h),$$
$$h=h(a). \tag{3.2}$$

Putting $F_n(\cdot,h)=\phi_\sigma(\cdot)+\Delta(\cdot)$ (where $\sigma=\sigma(h)$) we represent the last integral as the sum of two integrals with respect to the Gaussian measure ϕ_σ and the signed measure Δ. Replacing $F_n(\cdot, h)$ on ϕ_σ in (3.2) we have the following integral

$$I_\phi = (2\pi)^{-k/2} \det{}^{-1/2} \sigma(h) \int_{\mathscr{E}-\bar{a}} \exp\left\{n\Psi\left(\frac{x}{\sqrt{n}}\right)\right\} dx,$$

where

$$\Psi(h) = \Lambda(h, a) - (h, v) - \tfrac{1}{2}(\sigma^{-1}(h)\, v, v), \qquad h = h(v).$$

It is easy to prove that

$$\det{}^{-1/2} \sigma(h) = \det{}^{-1/2} U \left(1 + \theta_3 \frac{l}{\sqrt{n}}\right), \qquad |\theta_3| < c_7.$$

Using the Taylor expansion at the point $v = a$ from Lemma 2 we find that

$$\begin{aligned} \Lambda(h(v+a), v+a) &= \Psi(v) + \theta_4(U^{-1}v, v)\, \gamma^{-2} \max(|v+a|, |v|) \\ &= \Psi(v) + \theta_5 n^{-3/2}, \quad |\theta_s| < c_8, \quad s = 4, 5, \end{aligned} \tag{3.3}$$

if $v + a \in (1/\sqrt{n})\, K$. Therefore $I_\phi = \mu_n(\mathscr{E})(1 + \rho_1)$, where

$$|\rho_1| \leqslant \left(1 + c_7 \frac{l}{\sqrt{n}}\right) \exp\{c_8 n^{-1/2}\} - 1 \leqslant c_9 \frac{l}{\sqrt{n}}.$$

Now we consider the integral

$$I_\Delta = \int_{\mathscr{E}-\bar{a}} \exp\{-\sqrt{n}\,(h, x)\}\, \Delta(dx).$$

Put

$$\begin{aligned} &\mathscr{E}(t) = (\mathscr{E} - \bar{a}) \cap \{x: -\sqrt{n}(h, x) > t\}, \\ &\Delta(t) = \Delta(\mathscr{E}(t)), \qquad t \in R_1, \\ &\quad \alpha = \inf\{-\sqrt{n}\,(h, x): x + \bar{a} \in \mathscr{E}\}, \\ &\quad \beta = \sup\{-\sqrt{n}\,(h, x): x + \bar{a} \in \mathscr{E}\}. \end{aligned}$$

We have

$$|I_\Delta| = \left|\int_\alpha^\beta \exp\{t\}\, d\Delta(t)\right| \leqslant 2 \exp\{\beta\} \sup\{|\Delta(t)|: t \in R_1\}.$$

Since the diameter of the cube K is $\leqslant \sqrt{\gamma}\, l^{-1}$, it follows from Lemma 1 that $|\beta| < c_{10}$. By Lemma 4 with $W_n(\cdot) = F_n(\cdot, h)$ we have that for all $t \in R_1$

$$|\Delta(t)| < c_{11} \det^{-1/2} U \cdot \nu(K) \frac{l}{\sqrt{n}} \gamma^{-3/2} \exp\{c_{12}\gamma^{-3/2}\}.$$

Here we used the trivial inequalities

$$\nu((\mathscr{E}(t))_\varepsilon) \leqslant \nu(K_\varepsilon) \leqslant (f + 2\varepsilon)^k \leqslant \nu(K) \exp\{4k\varepsilon f^{-1}\},$$
$$\nu((\partial\mathscr{E}(t))_{2\varepsilon}) \leqslant \nu((\partial K)_{2\varepsilon}) \leqslant c_{13}\varepsilon f^{-1}\nu(K) \exp\{8k\varepsilon f^{-1}\},$$

where $\varepsilon = c_4\rho(\gamma n)^{-1}$ and $\varepsilon f^{-1} \leqslant c_{14}\gamma^{-3/2}\, \ln^{-1/2}$. We remark that by (3.3)

$$n\Lambda(h(v+a), v+a) \geqslant n\Lambda(h(a), a) - c_{15}, \qquad v + a \in \frac{1}{\sqrt{n}} K,$$

and therefore

$$\nu(K) \leqslant (2\pi)^{k/2} \det^{1/2} U \exp\{-\Lambda(h(a), a) + c_{15}\}\, \mu_n(K).$$

Theorem 1 is an immediate consequence of (3.1). Really let $\{K_i\}$ be a partition of R_k on the cubes K_i with the rib of length f and let $\{\mathscr{E}_i = \mathscr{E} \cap K_i\}$ be the partition of the set $\mathscr{E}$. We obtain Theorem 1 applying (3.1) for each $\mathscr{E}_i$ and summarizing it on i.

4. Large Deviations on Outside of Balls

We apply Theorem 1 for the case when $\mathscr{E}$ is the outside of the ball centered at zero. Assume that (B_1) the maximal eigenvalue of the covariance matrix U is simple.

Let γ_1 be the maximal eigenvalue of the covariance matrix U and e_1 the corresponding eigenvector of U, $|e_1| = 1$. Put

$$\begin{gathered} b^{\pm}(\tau) = \sup\{\Lambda(h(v), v) \colon |v| = \tau, \qquad \pm(v, e_1) \geqslant 0\}, \quad \tau > 0, \\ \lambda^{\pm}(\tau) = \tau^2(2\gamma_1)^{-1} + b^{\pm}(\tau), \end{gathered} \tag{4.1}$$

Theorem 2. *If* (A) *and* (B_1) *hold, then*

$$F_n(S^c(r)) = \frac{1}{2}\phi_U(S^c(r)) \left[\exp\left\{n\lambda^+\left(\frac{r}{\sqrt{n}}\right)\right\} + \exp\left\{n\lambda^-\left(\frac{r}{\sqrt{n}}\right)\right\}\right][1 + O(\sqrt{r}\, n^{-1/4})]$$

uniformly on $r \in [1, o(\sqrt{n})]$.

From a result of Zolotarev [10] it follows that if (B_1) holds, then

$$\phi_U(S^c(r)) = K_U r^{-1} \exp\{-r^2(2\gamma_1)^{-1}\}(1+o(1)), \qquad r \to \infty, \tag{4.2}$$

where the constant K_U depends on the eigenvalies of the matrix U.

For the case of the symmetric distribution of ξ we have $\lambda^+(\tau) = \lambda^-(\tau) = \lambda(\tau)$.

Consequence. If (A) and (B_1) are satisfied and ξ has a symmetric distribution, then

$$F_n(S^c(r)) = \phi_U(S^c(r)) \exp\left\{n\lambda\left(\frac{r}{\sqrt{n}}\right)\right\} [1 + O(\sqrt{r}\, n^{-1/4})]$$

uniformly on $r \in [1, o(\sqrt{n})]$.

Proof of Theorem 2. First, we investigate the asymptotic behaviour of the distribution F_n on the set

$$\mathscr{E}^+ = [S(\sqrt{r}\, n^{1/4}) \backslash S(r)] \cap R_k^+,$$

where $R_k^+ = \{x \in R_k : (x, e_1) \geqslant 0\}$. The maximum of the continuous functional $\Lambda(h(v), v)$ on the compact set $A_\tau^+ = \{v \in R_k^+ : |v| = \tau\}$ in (4.1) correspond to some point $v^+ = v^+(\tau)$. Using Taylor's expansion at the point $v = 0$ and Lemma 2 we have

$$\Lambda(h(v), v) = -\tfrac{1}{2}(U^{-1}v, v) + O(|v|^3), \qquad |v| \to 0. \tag{4.3}$$

The maximum of the first summand in (4.3) on the set A_τ^+ is equal to $-\tau^2(2\gamma_1)^{-1}$ and it is achieved for $v = \tau e_1$. Thus

$$b^+(\tau) = -\tau^2(2\gamma_1)^{-1} + O(\tau^3), \qquad \tau \to 0. \tag{4.4}$$

Comparing (4.3) and (4.4) we find that

$$(U^{-1}v^+, v^+) = (U^{-1}e_1, e_1)\,\tau^2 + O(\tau^3).$$

It follows from here that

$$|v^+(\tau) - \tau e_1|^2 = O(\tau^3), \qquad \tau \to 0. \tag{4.5}$$

Put $\tau = r/\sqrt{n}$ and $\mathrm{h}^+ = h(v^+(\tau))$. Note that $h = h(v)$ is the gradient of the functional $\Lambda(h(v), v)$ (see Lemma 2) and the point of the maximum v^+ is not a border one for the set A_τ^+; therefore the vector h^+ has the same direction as the vector v^+. Using the following Taylor's expansion

$$\Lambda(h(v), v) = b^+(\tau) - (h^+, v - v^+) - \tfrac{1}{2}(U^{-1}(v - v^+), v - v^+) + o(\tau)\,|v - v^+|^2, \qquad v \in S(\varepsilon_1 \gamma^2),$$

we obtain that

$$\mu_n(\mathscr{E}^+) = (2\pi)^{-k/2} \det{}^{-1/2} U \exp\{nb^+(\tau)\}\, Q, \tag{4.6}$$

where

$$Q = \int_{\mathscr{E}^+} \exp\{q\}\, dx, \qquad x^+ = v^+ \sqrt{n}, \tag{4.7}$$

$$q = -\sqrt{n}\,(h^+, x - x^+) - \tfrac{1}{2}(U^{-1}(x - x^+), x - x^+) + O(\tau)\,|x - x^+|^2. \tag{4.8}$$

In order to investigate the integral Q we apply the substitution $x = Dy$ with the orthogonal matrix D. The matrix D may be choosed in such a way that $Dv^+ = \tau e_1$ and $\|D - I\| = O(\sqrt{\tau})$, $\tau \to 0$, in view of (4.5) (here I is unity matrix in R_k and $\|\cdot\|$ denotes the matrix-norm). Putting $y = Dx$ we have

$$q = -\sqrt{n}\,|h^+|\,(e_1, y - re_1) - \tfrac{1}{2}(U^{-1}(y - re_1), y - re_1)$$
$$+ O(\sqrt{\tau})\,|y - re_1|^2, \qquad y \in D\mathscr{E}^+,$$

if n is sufficiently large. It follows from the expansion (2.3) and (4.5) that

$$|h^+| = |U^{-1}v^+| + O(|v^+|^2) = \tau\gamma_1^{-1} + O(\tau^{-3/2}), \qquad \tau \to 0.$$

We have

$$-r\gamma_1^{-1}(e_1, y - re_1) - \tfrac{1}{2}(U^{-1}(y - re_1), y - re_1)$$
$$= \tfrac{1}{2}(\gamma_1^{-1} r^2 - (U^{-1}y, y)) \leqslant 0, \qquad y \in S^c(r)$$

and

$$|y - re_1|^2 \leqslant 2\,|r^2 - \gamma_1(U^{-1}y, y)|, \qquad y \in D\mathscr{E}^+,$$

if n is sufficiently large. From here we find that

$$Q = \int_{D\mathscr{E}^+} \exp\{\tfrac{1}{2}[\gamma_1^{-1}r^2 - (U^{-1}y, y)](1 + O(\sqrt{\tau}))\}\, dy$$
$$= \int_{D\mathscr{E}^+} \exp\{\tfrac{1}{2}[\gamma_1^{-1}r^2 - (U^{-1}y, y)]\}\, dy\, (1 + O(\sqrt{\tau}))$$
$$= (2\pi)^{k/2} \det{}^{1/2} U \exp\{r^2(2\gamma_1)^{-1}\}\, \phi_U(D\mathscr{E}^+)(1 + O(\sqrt{\tau})). \tag{4.9}$$

It is easy to prove that

$$\phi_U(\mathscr{E}^+ \circ (D\mathscr{E}^+)) = \phi_U(\mathscr{E}^+)\, O(\sqrt{\tau}) \tag{4.10}$$

(the symbol $\circ$ denote the symmetrical difference of the sets).

From (4.6)–(4.10) it follows that

$$\mu_n(\mathscr{E}^+) = \frac{1}{2}\phi_U(\mathscr{E}^+)\exp\left\{n\lambda^+\left(\frac{r}{\sqrt{n}}\right)\right\}(1 + O(\sqrt{r}\,n^{-1/4}))$$

uniformly on $r \in [1, o(\sqrt{n})]$. The same asymptotic formula is true for $\mu_n((\mathscr{E}^+)_\varepsilon)$, $\varepsilon = \sqrt{\gamma}\,l^{-1}$.

LEMMA 6. *If* (A) *is satisfied, then*

$$F_n(S^c(r)) \leqslant 2\exp\{-r^2(4E\,|\xi|^2)^{-1}\}, \qquad r \in [0, \varepsilon_3 E\,|\xi|^2\sqrt{n}],$$

where the positive constant ε_3 *depends only on* m_0 *and* t_0.

This lemma is a consequence of more general results by Yurinskiĭ [9] (Lemmas 4.1 and 4.2).

By Lemma 6

$$F_n(S^c(\sqrt{r}\,n^{-1/4}) \leqslant 2\exp\{-\varepsilon_2 r\sqrt{n}\}, \qquad \varepsilon_2 > 0.$$

Finally, using Theorem 1 and (4.2) we have

$$F_n(S^c(r) \cap R_k^+) = \frac{1}{2}\phi_U(S^c(r))\exp\left\{n\lambda^+\left(\frac{r}{\sqrt{n}}\right)\right\}$$
$$\times (1 + O(\sqrt{r}\,n^{-1/4})), \qquad n \to \infty,$$

uniformly on $r \in [1, o(\sqrt{n})]$. This relation remains true if R_k^+ and λ^+ are replaced by R_k^- and λ^-. Theorem 2 follows.

In conclusion we shall formulate an analogue of Theorem 2 when (B_2) the maximal eigenvalue of the covariance matrix U is multiple, $\gamma_1 = \cdots = \gamma_l > \gamma_{l+1} \geqslant \cdots \geqslant \gamma_k$, $1 < l \leqslant k$ (where γ_i, $1 \leqslant i \leqslant k$, is the eigenvalues of U).

Let M_l be the invariant subspace of U corresponding to the maximal eigenvalue and let $M(w)$ be the one-dimensional subspace spanning on the vector $w \in R_k$. Put

$$b(\tau, w) = \sup\{\Lambda(h(v), v): |v| = \tau, v \in M_l^\perp \oplus M(w)\},$$

where $\tau > 0$, $w \in M_l$, $\perp$ and $\oplus$ denote respectively the orthogonal addition to R_k and the product of the orthogonal subspaces. Put

$$\lambda(\tau, w) = \tau^2(2\gamma_1)^{-1} + b(\tau, w), \qquad \tau > 0, \quad w \in M_l.$$

THEOREM 3. *If* (A) *and* (B_2) *hold, then*

$$F_n(S^c(r)) = \phi_U(S^c(r)) \frac{1}{\varphi_l} \int_V \exp\left\{n\lambda\left(\frac{r}{\sqrt{n}}, w\right)\right\} ds \times [1 + O(\sqrt{r}\, n^{-1/4})]$$

uniformly on $r \in [1, o(\sqrt{n})]$. *Here* $V = \{w \in M_l \colon |w| = 1\}$ *is the surface of the l-dimensional ball, ds is the element of the surface V and* $\varphi_l = 2\pi^{1/2}[\Gamma(\frac{1}{2})]^{-1}$.

For the case $l = k$, Theorem 3 was proved by Richter [7].

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